Exam IV
Section I
Part A — No Calculators

1. A p. 73

$$\lim_{x\to 0} \frac{\frac{1}{x-1}+1}{x} = \lim_{x\to 0} \frac{1+(x-1)}{x(x-1)} = \lim_{x\to 0} \frac{x}{x(x-1)} = \lim_{x\to 0} \frac{1}{x-1} = -1$$

2. E p. 73

Since $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$, the integrand has the form $\int e^u\,du$.

Thus $\int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx = \int e^{\sqrt{x}} \frac{1}{2\sqrt{x}}\,dx = e^{\sqrt{x}} + C$

3. D p. 74

$$y = \frac{3}{4+x^2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-3\cdot 2x}{(4+x^2)^2} = \frac{-6x}{(4+x^2)^2}$$

4. B p. 74

By the Second Fundamental Theorem, $\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$.

Then $F(x) = \int_1^x (\cos 6t + 1)\,dt \quad \Rightarrow \quad F'(x) = \cos 6x + 1$.

5. C p. 74

Differentiating $x + xy + 2y^2 = 6$ implicitly gives $1 + y + x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$.

Evaluating at (2,1) gives: $6\frac{dy}{dx} = -2$

$$\frac{dy}{dx} = -\frac{1}{3}.$$

6. C p. 75

This limit is the definition of the derivative of $f(x) = 3x^5$ at $x = \frac{1}{2}$.

Since $f'(x) = 15x^4$, $f'\left(\frac{1}{2}\right) = 15\cdot\left(\frac{1}{2}\right)^4 = \frac{15}{16}$.

7. E p. 75

The slope of the given line $5x - y + 6 = 0$ is $m = 5$.
Since the tangent to the graph of $p(x)$ at $x = 4$ is parallel to that given line, we know that $p'(4) = 5$.

$p(x) = (x-1)(x+k) \Rightarrow p'(x) = (x-1) + (x+k)$
$\Rightarrow p'(4) = (4-1) + (4+k) = 7 + k$

Since $p'(4)$ must equal 5, we have $7 + k = 5$.
Hence $k = -2$.

8. A p. 75

$\cos x = e^y \Rightarrow y = \ln(\cos x)$
$\Rightarrow \dfrac{dy}{dx} = \dfrac{-\sin x}{\cos x} = -\tan x$

9. D p. 76

$a(t) = 12t^2$
$v(t) = \int a(t)\,dt = 4t^3 + C$
$v(0) = 6 \Rightarrow C = 6 \Rightarrow v(t) = 4t^3 + 6$
$s(t) = \int v(t)\,dt = t^4 + 6t + D$
$s(2) - s(0) = [16 + 12 + D] - [0 + 0 + D] = 28$

10. B p. 76

$A = x^2$
$\dfrac{dA}{dt} = 2x\dfrac{dx}{dt}$

Since we are given that $\dfrac{dA}{dt} = 3\dfrac{dx}{dt}$, we can substitute.

$3\dfrac{dx}{dt} = 2x\dfrac{dx}{dt} \Rightarrow x = \dfrac{3}{2}$

11. E p. 76

$$M = \frac{1}{e-1}\int_1^e \frac{1}{x}\,dx = \frac{1}{e-1}\cdot \ln e = \frac{1}{e-1}$$

12. B p. 77

$$\lim_{x\to\infty} \frac{3x^2+1}{(3-x)(3+x)} = \lim_{x\to\infty} \frac{3x^2+1}{9-x^2}$$
$$= \lim_{x\to\infty} \frac{3 + \frac{1}{x^2}}{\frac{9}{x^2} - 1} = -3$$

13. D p. 77

$$\int_{-2}^{2} (x^7 + k)\,dx = \int_{-2}^{2} x^7\,dx + \int_{-2}^{2} k\,dx$$

Note that $y = x^7$ is an odd function, so $\int_{-2}^{2} x^7\,dx = 0$.

Thus $\int_{-2}^{2} (x^7 + k)\,dx = 0 + 4k$. Since we are given that the value of the definite integral is 16, we conclude that $k = 4$.

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14. D p. 77

$f(x) = \dfrac{\tan x}{\sin x} = \dfrac{1}{\cos x}$ for $x \neq \pi$.

For f to be continuous at $x = \pi$, the value $f(\pi)$ must be defined in such a way so that $f(\pi) = \lim\limits_{x \to \pi} f(x)$.

$\lim\limits_{x \to \pi} f(x) = \lim\limits_{x \to \pi} \dfrac{1}{\cos x} = \dfrac{1}{\cos \pi} = -1$.

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15. C p. 78

$$\begin{aligned} f(x) &= x^4 - 18x^2 \\ f'(x) &= 4x^3 - 36x \\ &= 4x(x^2 - 9) \\ &= 4x(x+3)(x-3) \end{aligned}$$

$\Rightarrow$ The critical numbers are at $x = 0, \pm 3$.

$$f''(x) = 12x^2 - 36$$

Thus: $\begin{cases} f''(0) = -36 < 0 & \text{Relative maximum at } x = 0 \\ f''(-3) = 108 - 36 > 0 & \text{Relative minimum at } x = -3 \\ f''(3) = 108 - 36 > 0 & \text{Relative minimum at } x = 3 \end{cases}$

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16. E p. 78

$$\begin{aligned} y &= 3x^5 - 10x^4 \\ y' &= 15x^4 - 40x^3 \\ y'' &= 60x^3 - 120x^2 \\ &= 60x^2(x-2) \end{aligned}$$

The second derivative changes sign at $x = 2$, but **not** at $x = 0$. Hence the only inflection point is at $x = 2$.

17. D p. 78

Since $h(x) = f(g(x))$, we have $h'(x) = f'(g(x)) \cdot g'(x)$ by the Chain Rule.
The graph of h has a horizontal tangent line if $h'(x) = 0$.
Hence we need $f'(g(x)) \cdot g'(x) = 0$.
This occurs if $f'(g(x)) = 0$ or if $g'(x) = 0$.

Since $f'(-2)$ and $f'(1)$ both have the value 0, the first condition is satisfied if $g(x)$ is either -2 or 1.

$g(x) = -2$ if $x = -4$ or $x = -2$.
$g(x) = 1$ if $x = 0$ or $x = 3.4$.

There are 4 horizontal tangents from this condition.

The second condition is satisfied at each horizontal tangent point for the function g. These are at $x = -3$, $x = 0$, and $x = 2$.

The total list of x-values at which there are horizontal tangents is:
$x = -4, -3, -2, 0, 2, 3.4$. There are 6 places where this happens.

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18. D p. 79

Cross sections taken perpendicular to the y-axis on the interval $[0, \frac{\pi}{2}]$ are circular. The radius of each circular cross section is an x-coordinate.
Since $y = \arcsin x$, we have $x = \sin y$.

Thus the volume is computed by $V = \pi \int_0^{\pi/2} (\sin y)^2 \, dy$.

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19. A p. 79

$y = xe^{-kx}$
$y' = e^{-kx} - kxe^{-kx} = e^{-kx}(1-kx) \quad \Rightarrow \quad$ Critical number at $x = \frac{1}{k}$.

$y(\frac{1}{k}) = \frac{1}{k} \cdot e^{-1} \quad \Rightarrow$
The point on the curve is $\left(\frac{1}{k}, \frac{1}{ke}\right)$.

To verify that there really is a maximum value at $x = \frac{1}{k}$, use the Second Derivative Test.
$y'' = e^{-kx}(-k) - ke^{-kx}(1 - kx) = -ke^{-kx}(2 - kx)$
Then $y''(\frac{1}{k}) = -k e^{-1}(1)$. Since k is given to be positive, $y''(\frac{1}{k}) < 0$. Hence there is a maximum value at $x = \frac{1}{k}$.

20. D p. 79

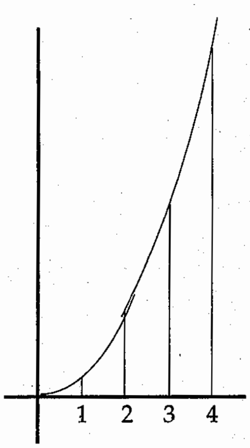
The slopes of the segments in the slope field depend only upon the variable y. We can tell this because for a given y, the slopes of the segments do not change as x varies. Therefore we can eliminate the three suggested answers that have $\frac{dy}{dx}$ depend upon x. For $y = 0$, the slopes are positive. The correct answer cannot be $\frac{dy}{dx} = y - 2y^2$, for that would have a slope of 0 when $y = 0$.

21. B p. 80

$y' > 0 \quad \Rightarrow \quad y$ is an increasing function.
$y'' < 0 \quad \Rightarrow \quad y$ is concave down.

The only increasing and concave down curve is (B).

22. C p. 80



With $n = 3$, the width of each subinterval is 1.

Thus $T_3 = \frac{1}{2}\left[1 + 2\cdot 4 + 2\cdot 9 + 16\right] = \frac{43}{2}$.

23. D p. 80

$f(x) = 4x^3 - 21x^2 + 36x - 4$

For the graph of f to be deceasing, it is necessary that $f'(x) < 0$.

$$f'(x) = 12x^2 - 42x + 36 < 0 \quad \Rightarrow \quad 2x^2 - 7x + 6 < 0$$
$$\Rightarrow \quad (2x-3)(x-2) < 0$$
$$\Rightarrow \quad \frac{3}{2} < x < 2$$

For the graph of f to be concave up, it is necessary that $f''(x) > 0$.

$$f''(x) = 24x - 42 > 0 \quad \Rightarrow \quad x > \frac{7}{4}$$

We must have <u>both</u> of these inequalities true. Hence $\frac{7}{4} < x < 2$.

24. A p. 81

Since the rate of growth is $1500\,e^{3t/4}$, we start with $\frac{dx}{dt} = 1500\,e^{3t/4}$.

Then $dx = 1500\,e^{3t/4}\,dt$.

$x = 2000\,e^{3t/4} + C$

Knowing that $x = 2000$ when $t = 0$ allows us to evaluate C:

$2000 = 2000\,e^0 + C \quad \Rightarrow \quad C = 0.$

Hence $x = 2000\,e^{3t/4}$.

When $t = 4$, we have $x = 2000\,e^3$.

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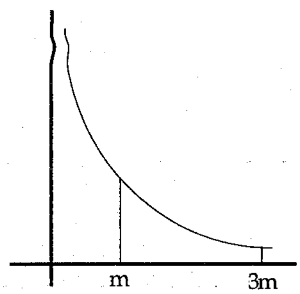
25. A p. 81



$$\int_m^{3m} \frac{1}{x}\,dx = \ln|x|\Big]_m^{3m} = \ln(3m) - \ln m = \ln 3.$$

This result is independent of m.

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26. C p. 82

$$x(t) = \ln t + \frac{t^2}{18} + 1 \quad \Rightarrow \quad v(t) = x'(t) = \frac{1}{t} + \frac{t}{9}$$

$$\Rightarrow \quad a(t) = x'(t) = -\frac{1}{t^2} + \frac{1}{9}.$$

The acceleration is zero when $t = 3$.

Then $v(3) = \frac{1}{3} + \frac{3}{9} = \frac{2}{3}$.

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27. D p. 82

$$\int 6\sin x\cos^2 x\,dx = -6\int (\cos x)^2(-\sin x)\,dx$$

$$= -6\frac{\cos^3 x}{3} + C = -2\cos^3 x + C$$

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28. D p. 82

By the Second Fundamental Theorem, $G'(x) = \sin(\ln 2x)$.

Then $G''(x) = \cos(\ln 2x)\cdot\frac{1}{2x}\cdot 2 = \frac{\cos(\ln 2x)}{x}$.

Hence $G''\left(\frac{1}{2}\right) = \frac{\cos(\ln 1)}{1/2} = 2\cos 0 = 2$.

Exam IV
Section I
Part B — Calculators Permitted

1. D p. 83

   I. $f''(x)$ changes sign at $x = -1$. True
   II. $f''(x) < 0$ on the interval $(-1,3)$. True
   III. Since $\dfrac{d(f'(x))}{dx} < 0$ in the vicinity of $x = 1$, the function f' is decreasing at $x = 1$.
   False

2. D p. 83

$$\begin{aligned}\frac{\ln x^2 - x\ln x}{x-2} &= \frac{2\ln x - x\ln x}{x-2}\\ &= \frac{(2-x)\ln x}{x-2}\\ &= -\ln x \quad \text{for } x \neq 2\end{aligned}$$

Since f is given to be continuous at $x = 2$, $\lim\limits_{x\to 2} f(x) = f(2)$.

But $\lim\limits_{x\to 2} f(x) = -\ln 2$. Hence $f(2) = -\ln 2$.

3. E p. 84

Condition		Interpretation	Points
$f(x) > 0$	$\Rightarrow$	the point $(x,f(x))$ is above the x-axis.	M, P, Q, R
$f'(x) < 0$	$\Rightarrow$	f is decreasing.	M, R
$f''(x) < 0$	$\Rightarrow$	the graph of f is concave down.	Q, R

All **three** conditions occur only at point R.

4. D p. 84

With the substitution $u = \sqrt{1+x}$, we have $x = u^2 - 1$ and $dx = 2u\,du$.

Then $\int 60x\sqrt{1+x}\,dx = \int 60\,(u^2-1)\cdot u \cdot 2u\,du$

$= \int (120u^4 - 120u^2)\,du = 24u^5 - 40u^3 + C$

5. B p. 84

$$f'(x) = \frac{2x}{2\sqrt{x^2 + .0001}}.$$

   I. Since $f'(0)$ exists, f is continuous at $x = 0$. False
   II. $f'(0) = 0$. Thus there is a horizontal tangent at $x = 0$. True
   III. $f'(x)$ is defined as above with $f'(0) = 0$. False

6. C p. 85

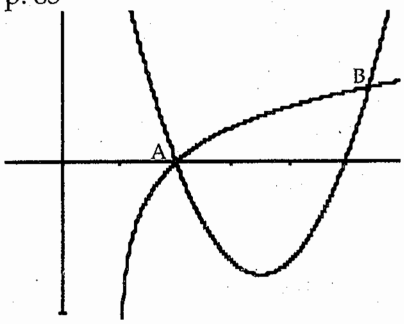
$f(x) = \dfrac{1}{x^2} = x^{-2}$ $\qquad g(x) = \arctan x$

$f'(x) = -\dfrac{2}{x^3}$ $\qquad g'(x) = \dfrac{1}{1+x^2}$

When $\dfrac{1}{1+x^2} = \dfrac{-2}{x^3}$, then $x^3 = -2 - 2x^2$.

Solving $x^3 + 2x^2 + 2 = 0$ gives $x = -2.359$.

7. B p. 85



The curves intersect at
A, where $x = 2$
and B, where $x = 5.4337$.

Then the area is:

$$A = \int_2^{5.4337} (g(x) - f(x))\,dx \approx 7.36 .$$

8. B p. 85

The average rate of change of a function f over an interval $[a, b]$ is defined to be $\dfrac{f(b) - f(a)}{b - a}$.

With the function $f(x) = \displaystyle\int_0^x \sqrt{1 + \cos(t^2)}\,dt$ and the interval $[1, 3]$,

we have, first of all, $f(3) - f(1) = \displaystyle\int_0^3 \sqrt{1+\cos(t^2)}\,dt - \int_0^1 \sqrt{1+\cos(t^2)}\,dt$

$$= \int_1^3 \sqrt{1+\cos(t^2)}\,dt .$$

Hence the average rate of change of the function is $\dfrac{1}{2}\displaystyle\int_1^3 \sqrt{1+\cos(t^2)}\,dt \approx 0.86.$

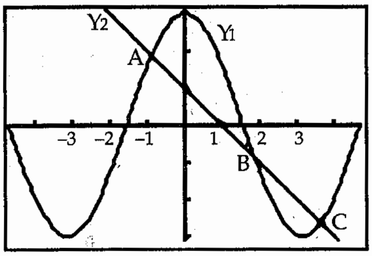
9. D p. 86

I. $f'(0) = 1 \Rightarrow$ f is increasing at $x = 1$. False

II. $f'(x) > 0$ for $x < 2$ and $f'(x) < 0$ for $x > 2$.
Hence f is increasing to the left of $x = 2$ and decreasing to the right of $x = 2$. There is a relative maximum there. True

III. $f'(x)$ is increasing on an open interval containing $x = -1$.
Hence, the graph is concave up at $x = -1$. True

10. D p. 86



The graphs are shown to the left. The points A, B, and C have x-coordinates as follows:

A = −0.88947; B = 1.86236; C = 3.63796

Use a calculator to evaluate

$$\int_A^B (Y1 - Y2)\,dx + \int_B^C (Y2 - Y1)\,dx$$

The value is approximately 4.98.

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11. C p. 86
Parts of the rectangles are above the curves for B and C.
The trapezoids are all on or under the curves for A, C, and E.
Hence, the answer is C.

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12. B p. 87

$V = \frac{4}{3}\pi r^3 \quad \Rightarrow \quad \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

At t = 6 seconds, $\frac{dV}{dt}$ can be estimated from the graph. The slope of the tangent to the curve at $t = 6$ is approximately $\frac{2\pi}{2} = \pi\ \text{in}^3/\text{sec}$.

Hence we use $\frac{dV}{dt} = \pi$.

In addition, when $t = 6$, $V = 4\pi$. This allows us to find the radius r when $t = 6$. From the original volume formula,

$\frac{4}{3}\pi r^3 = 4\pi \quad \Rightarrow \quad r^3 = 3 \quad \Rightarrow \quad r = 3^{1/3}.$

Now from the formula for the rate of change of the volume, we have:

$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \Rightarrow \quad \pi = 4\pi(3^{1/3})^2 \frac{dr}{dt}.$ Hence $\frac{dr}{dt} = \frac{1}{4 \cdot 3^{2/3}} \approx 0.12$

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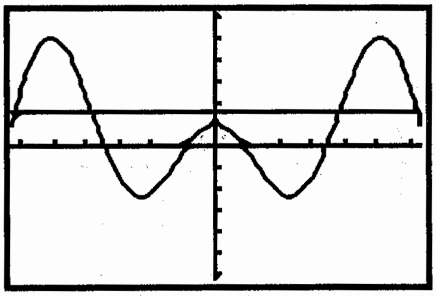
13. B p. 87

$y = x\cos x$
$y' = \cos x - x\sin x$

We want $\cos x - x\sin x = \frac{\pi}{2}$.
Shown to the right are graphs of
$Y1 = \cos x - x\sin x$
and $Y2 = \frac{\pi}{2}$.
The viewing window is:
$-2\pi \le x \le 2\pi$; $-6.2 \le y \le 6.2$.
Y1 and Y2 intersect four times on the interval $[-2\pi, 2\pi]$.



14. C p. 88

If we multiply out the given expression, we can obtain a simplified version:

$$x^2 - 2xy + y^2 = y^2 - xy$$
$$x^2 = xy$$

Differentiate implicitly: $2x = x\frac{dy}{dx} + y$

Then $\frac{dy}{dx} = \frac{2x-y}{x}$.

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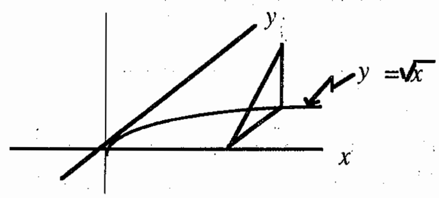
15. B p. 88

Since the cross sections perpendicular to the x-axis are isosceles right triangles, we want to create an integral of the form $\int_a^b f(x)\,dx$ for this volume.



A leg of the triangle is a typical y-coordinate and the cross sectional area is

$\frac{1}{2}yy = \frac{1}{2}y^2$ where $y = \sqrt{x}$. Hence, the volume is $V = \int_0^4 \frac{1}{2}x\,dx = \left.\frac{x^2}{4}\right|_0^4 = 4$.

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16. D p. 89

Since the graph of f is made up of straight line segments and a semicircle, we can evaluate F at each of the integer coordinates from $x = 0$ through $x = 8$. Note that sections of the graph below the t-axis decrease the value of F.

x	0	1	2	3	4	5	6	7	8
F(x)	0	.25	1	1.5	.715	–.071	.429	.929	.429

The value of $F(x)$ changes from positive to negative at an x-coordinate between 4 and 5.
The value of $F(x)$ changes from negative to positive at an x-coordinate between 5 and 6.
Since $f(t) \ge 0$ for $0 \le t \le 3$ and for $5 \le t \le 7$, F is increasing on those intervals.
Since $f(t) \le 0$ for $3 \le t \le 5$ and for $7 \le t \le 8$, F is decreasing on those intervals.
Hence the only zeros for the function F occur in the intervals $[4,5]$ and $[5,6]$.

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17. B p. 89

Separate variables to solve the differential equation.

$$\frac{dN}{dt} = 2N \Rightarrow \frac{dN}{N} = 2dt$$
$$\Rightarrow \ln N = 2t + C$$
$$\Rightarrow N = e^{2t+C} = e^{2t}\cdot e^{C} = De^{2t} \quad (\text{where } D = e^{C})$$

Since $N = 3$ when $t = 0$, we have $3 = De^0$, which implies $D = 3$.

Thus we can write: $N = 3e^{2t}$. Now let $N = 1210$.

Then we have $1210 = 3e^{2t} \Rightarrow \frac{1210}{3} = e^{2t}$

$$\Rightarrow 2t = \ln\frac{1210}{3}$$
$$\Rightarrow t = \frac{\ln 1210 - \ln 3}{2} \approx 3$$

Exam IV
Section II
Part A — Calculators Permitted

1. p. 91

(a) $\frac{dy}{dt} = -0.1(y-70) \Rightarrow \frac{dy}{y-70} = -0.1\,dt$

$\Rightarrow \ln(y-70) = -0.1t + C \quad (y \geq 70$ as the tea cools)

$y = 180$ when $t = 0$ gives $\ln(180-70) = -0.1 \cdot 0 + C$, so $C = \ln 110$.

Then we have
$$\ln(y-70) = -0.1t + \ln 110$$
$$y - 70 = e^{-0.1t + \ln 110} = e^{-0.1t} \cdot e^{\ln 110}$$
$$y = 70 + 110\,e^{-0.1t}$$

5: { 1: separates variables; 1: antiderivatives; 1: constant of integratio; 1: uses initial condition; 1: solves for y }

(b) When $t = 10$, we have $y = 70 + 110\,e^{-0.1 \cdot 10} = 10 + 110\,e^{-1} \approx 110.467°$.

1 : answer

(c) Determine t so that $y = 120$.

$120 = 70 + 110\,e^{-0.1t} \Rightarrow 50 = 110\,e^{-0.1t} \Rightarrow \frac{5}{11} = e^{-0.1t}$

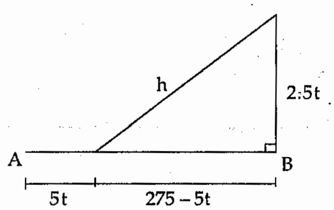
$\Rightarrow \ln\frac{5}{11} = -0.1t \Rightarrow t = \frac{\ln 5 - \ln 11}{-0.1} \approx$ 7.885

The tea is safe to drink after 7.885 minutes .

3: { 1: $y(t) = 120$; 1: solves for t; 1: answer }

2. p. 92

(a)



Let t be the number of seconds after the balloon is released (and the cat starts running). The variable h denotes the distance between the cat and the balloon.

Then the diagram to the left shows the important distances.

$$h^2 = (275-5t)^2 + (2.5t)^2$$
$$2h\frac{dh}{dt} = 2(275-5t)(-5) + 2(2.5t)(2.5) \Rightarrow \frac{dh}{dt} = \frac{(275-5t)(-5) + (2.5t)(2.5)}{h}$$

When $t = 40$, the triangle has legs of 75 feet and 100 feet. Hence the hypotenuse $h = 125$ feet.

Then $\frac{dh}{dt} = \frac{(75)(-5) + (100)(2.5)}{h} = \frac{-375 + 250}{125} = -1.$

The distance between the cat and the balloon is decreasing at 1 ft/sec.

5: { 1: expression for distance; 1: $\frac{dh}{dt}$; 1: answer; 1: rate }

(b) When $t = 50$, the triangle has legs of 25 feet and 125 feet. Then the hypotenuse $h = 25\sqrt{26}$ feet. Then $\frac{dh}{dt} = \frac{(25)(-5) + (125)(2.5)}{25\sqrt{26}} = \frac{-5 + 12.5}{\sqrt{26}}$ $= \frac{7.5}{\sqrt{26}} \approx 1.471$. Now distance between the cat and balloon is increasing at a rate of about 1.471 ft/sec.

2: { 1: $\frac{dh}{dt}$ at $t = 50$; 1: explanation }

(c) Using the expression for $\frac{dh}{dt}$ from part (a) and multiplying we obtain

$\frac{dh}{dt} = \frac{-1375 + 25t + 6.25t}{h}$. $\frac{dh}{dt} = 0$ when $31.25t = 1375$ and $t = 44$ sec

Since $\frac{dh}{dt}$ goes from neg to pos at $t = 44$, this is a relative minimum.

2: { 1: answer; 1: explanation }

3. p. 93

(a) The given function p' has degree 7. Therefore p has degree 8.

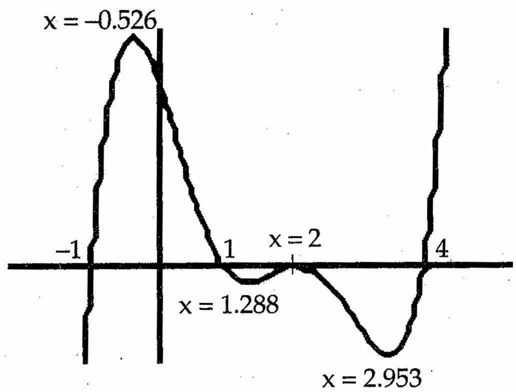
1 : answer

(b) The instantaneous rate of change of the function p at $x = 6$ is simply the value of $p'(6)$.

$p'(6) = 7 \cdot 5 \cdot 4^2 \cdot 2^3 = 4480.$

$2: \begin{cases} 1: p'(x) \\ 1: p'(6) \end{cases}$

(c) A graph of the given derivative function p' is shown below. There has been no attempt to get the vertical scale accurate. However, x-coordinates of the relative extreme values are shown, and the intervals of positive and negative values are correct.



The function p is increasing if $p'(x) > 0$. This occurs for

$-1 < x < 1$ and for $x > 4$.

$3: \begin{cases} 1: p'(x) > 0 \\ 2: \text{answer} \end{cases}$

(d) We know that the graph of p will be concave down on those intervals on which $p'(x)$ is decreasing or $p''(x)$ is negative.

$p''(x) = (x-1)(x-2)^2(x-4)^3 + (x+1)(x-2)^2(x-4)^3 +$

$2(x^2-1)(x-2)(x-4)^3 + 3(x^2-1)(x-2)^2(x-4)^2.$

Graphing p'' we see that the zeros of p'' are -0.526, 1.288, 2, and 2.953 and the graph of p is concave down on $-0.526 < x < 1.288$ and for $2 < x < 2.953$.

(On the graph of p'' at $x = 4$ there is a relative minimum, but not a zero.)

$3: \begin{cases} 1: p'(x) < 0 \\ 2: \text{answer} \end{cases}$

Exam IV
Section II
Part B — No Calculators

4. p. 94

(a) $\lim_{x\to 0^+} f(x) = 2$ and $\lim_{x\to 0^-} f(x) = 2$. Thus $\lim_{x\to 0} f(x) = 2$.	1: answer
(b) $\lim_{h\to 0^-} \frac{f(0+h)-f(0)}{h}$ = the left-hand slope at $x = 0$ $= 1$	1: answer
(c) To the left of $x = 0$, $f'(x)$ is constant (with a value of 1). Just to the right of $x = 0$, $f'(x)$ appears to have a value of about 5. Thus $\lim_{x\to 0} f'(x)$ does not exist.	$3: \begin{cases} 1: \lim_{x\to 0^-} f'(x) \\ 1: \lim_{x\to 0^+} f'(x) \\ 1: \text{conclusion} \end{cases}$
(d) $\int_{-1}^{0} f(x)\,dx = \frac{3}{2}$. This is obtained by counting areas in the graph.	2: answer
(e) $\int_{-2}^{4} f(x)\,dx \approx \frac{2}{2}[0+2\cdot 2+2\cdot 6+2] = 18$	$2: \begin{cases} 1: \text{Trapezoid method} \\ 1: \text{answer} \end{cases}$

5. p. 95

(a) Since $g(x) = \dfrac{x \cdot |x|}{x^2+1}$, then $g(-x) = \dfrac{-x \cdot |-x|}{(-x)^2+1} = \dfrac{-x \cdot |x|}{x^2+1} = -g(x)$ for all x.

Differentiate the equation obtained above: $g(-x) = -g(x)$.
$g'(-x) \cdot (-1) = -g'(x) \Rightarrow g'(-x) = g'(x)$

Hence the derivative of the function g is an even function.

2: answer

(b) For $x \geq 0$, $g(x) = \dfrac{x^2}{x^2+1}$

Then $g'(x) = \dfrac{(x^2+1)(2x) - x^2(2x)}{(x^2+1)^2} = \dfrac{2x}{(x^2+1)^2}$

Therefore $g'(2) = \dfrac{4}{25}$.

$2: \begin{cases} 1: g'(x) \\ 1: g'(2) \end{cases}$

(c)
$$\int_0^1 g(x)\,dx = \int_0^1 \frac{x^2}{x^2+1}\,dx$$
$$= \int_0^1 \frac{(x^2+1) - 1}{x^2+1}\,dx = \int_0^1 \left[1 - \frac{1}{x^2+1}\right] dx$$
$$= \left[x - \text{Arctan } x\right]_0^1$$
$$= (1 - \text{Arctan } 1) - (0 - \text{Arctan } 0) = 1 - \frac{\pi}{4}$$

$2: \begin{cases} 1: \text{antiderivative} \\ 1: \text{answer} \end{cases}$

(d) $\lim\limits_{x \to \infty} g(x) = \lim\limits_{x \to \infty} \dfrac{x^2}{x^2+1} = 1.$

1: answer

(e) For $x \geq 0$, $g(x) = \dfrac{x^2}{x^2+1}$ and $g'(x) = \dfrac{2x}{(x^2+1)^2}$.

On the interval $[0,\infty)$, $g'(x) \geq 0$, so g is an increasing function.
Since $g'(x)$ exists for all $x \geq 0$, the function g is continuous.

$g(0) = 0$ and $\lim\limits_{x \to \infty} g(x) = \lim\limits_{x \to \infty} \dfrac{x^2}{x^2+1} = 1.$

Since $g(x)$ never attains the value 1 (the numerator of the fraction is smaller than the denominator), the range of the function for $x \geq 0$ is the interval $[0,1)$.

As shown in part (a), g is an odd function. Then the values of the function for negative inputs x are the numerical opposites of the values for positive inputs x.
Hence the range of the function g is the open interval $(-1, 1)$.

$2: \begin{cases} 1: \text{answer} \\ 1: \text{justification} \end{cases}$

6. p. 96

(a) $G(x) = \int_{-4}^{x} f(t)\,dt$

1 : answer

Then $G(-4) = \int_{-4}^{-4} f(t)\,dt$. Provided the function f is defined at $t = -4$ (and it is), then this integral has a value of 0.

(b) By the Second Fundamental Theorem, $G'(x) = f(x)$.

Thus $G'(-1) = f(-1) = 2$.

$2: \begin{cases} 1: G'(x) = f(x) \\ 1: \text{answer} \end{cases}$

(c) The given graph is the derivative of G. The graph of G will be concave down if its derivative is decreasing; that is, if its second derivative is negative-valued. This occurs on the open intervals $(-4,-3)$ and $(-1,2)$.

$3: \begin{cases} 2: \text{intervals} \\ 1: \text{justification} \end{cases}$

(d) The maximum value of G occurs at either a critical point or an endpoint of the interval. G has critical points at $x = 1$ and $x = 3$, where $G'(x) = f(x) = 0$. By summing areas of regions we estimate that at the critical points

$$G(1) = \int_{-4}^{1} f(x)\,dx = 7 \quad \text{and} \quad G(3) = 7 - \frac{4}{3} = \frac{17}{3}.$$

$3: \begin{cases} 1: \text{answer} \\ 2: \text{justification} \end{cases}$

At the endpoints,

$$G(-4) = \int_{-4}^{-4} f(x)\,dx = 0 \text{ and } G(4) = \int_{-4}^{4} f(x)\,dx = 7 - \frac{4}{3} + 1 = \frac{20}{3}$$

Therefore, G has its maximum value of 7 at $x = 1$.